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The global controllability of some mechanical systems with absolutely elastic impacts☆

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Abstract

Sufficient conditions are presented for the stabilizability and global controllability of certain natural Lagrangian systems with a non-negative potential energy when there are ideal unilateral constraints. In the general case, the number of controls is less than the number of degrees of freedom and the controls are bounded by preassigned quantities. Examples of globally controlled systems with two degrees of freedom are considered in which the action of the unilateral constraints is modelled within the framework of classical collision theory.

1. The class of systems considered

The properties of stabilizability and controllability, which are understood in the traditional sense,[1] are investigated for natural Lagrangian systems. In a continuation of a previous analysis[2–4] the systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{u} \tag{1.1}$$

with a Lagrangian function

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}}/2 - B(\mathbf{q}), \quad \mathbf{q} \in \mathbf{T}^r \times \mathbf{R}^{n-r}$$

which is symmetric with respect to time reversal ($t \to -t$), where $\mathbf{T}^r$ is an r-dimensional torus, are discussed. The control $\mathbf{u} = (u_1, \ldots, u_n)^T$ is a vector function of the time t, which is summable in any finite interval and satisfies the constraints $|u_i| \le a_i$, where a_i are specified numbers ($i = 1, 2, \ldots, n$). Some of them may be zero, that is, the number of degrees of freedom n can exceed the number of controls. For example, if $a_i = 0$ ($i = 1, 2, \ldots, n$, $i \ne j$), then system (1.1) is "controlled using a scalar input u_j"

We further assume that ideal unilateral constraints

$$\mathbf{f}(\mathbf{q}) \ge \mathbf{0} \tag{1.2}$$

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are imposed on the system, where the smooth functions $f_i(\mathbf{q})(j=1, 2, \ldots, l)$ satisfy the conditions $\partial f_i/\partial\mathbf{q} \neq \mathbf{0}$ at the points $\mathbf{q}^*$ for which $f_i(\mathbf{q}^*)=0$, that is, regular surfaces serve as the boundaries of the sets (1.2) in the configurational space M. The phase space

$$TM = \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbf{T}^r \times \mathbf{R}^{2n-r}: \mathbf{f}(\mathbf{q}) \geq \mathbf{0}\} \tag{1.3}$$

The notation $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbf{T}^r \times \mathbf{R}^{2n-r}$ implies that the numerical values of the coordinates are taken from the corresponding covering space $\mathbf{R}^r \times \mathbf{R}^{2n-r}$.

We will assume that the potential energy $B(\mathbf{q})$ has a lower bound, that is, $B(\mathbf{q}) \geq 0$, $B(\mathbf{0})=0$ and the set of equilibrium positions

$$\zeta_0 = \{(\mathbf{q}, \dot{\mathbf{q}}) \in TM: \dot{\mathbf{q}} = \mathbf{0}, \mathbf{u} = \mathbf{0}\}$$

will therefore be a non-empty set. Remark 1 We will further assume that the minimum value of the functions $B(\mathbf{q})$ is reached at the point $\mathbf{q} \in M$, which does not at the one belong to two or more surfaces $f_i(\mathbf{q})=0(i \in 1, 2, \ldots, l)$.

If the separatrix surface in the space TM, the motion along which to a singular point occurs after an infinite time, corresponds to the inverse relation $\mathbf{u} = \mathbf{u}(\mathbf{q}, \dot{\mathbf{q}})$, then the surface is denoted by $\Omega(\mathbf{u}(\mathbf{q}, \dot{\mathbf{q}}))$.

We rose the problem of finding the sufficient conditions for global controllability, that is, of the possibility, in a finite time, of transferring object (1.1), subject to condition (1.2), from an arbitrary state $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \in TM$ to any preassigned state $(\mathbf{q}_f, \dot{\mathbf{q}}_f) \in TM$ after a finite time using admissible controls. The distinguishing feature is existence of unilateral constraints (1.2) which implies the possibility of impacts and makes the system substantially unsmooth.

Note that, in the neighbourhood of each surface $f_j(\mathbf{q})=0$, it is convenient to use a local description of the configuration using the coordinate substitution proposed earlier in Ref. 5. Furthermore, the vector $(s, \mathbf{y}^T)$, $\dim \mathbf{y} = n-1$ is introduced prior to the substitution, where $s=f_i(\mathbf{q})$, and no constraint is imposed on the remaining component of the configuration vector $\mathbf{y}$. If the surface $s=0$ is reached when $\dot{s} \neq 0$ during the motion of the system, an impact accuse, the consequences of which have to be determined. We shall confine ourselves to cases when the determination within the framework of classical theory of an absolutely elastic impact[6] in the notation adopted earlier[5]

$$\dot{s}_+ = -\dot{s}_-, \quad \mathbf{p}_+ = \mathbf{p}_-, \quad T_+ = T_- \tag{1.4}$$

does not contradict the mechanical nature of the object.

The minus and plus subscripts correspond to quantities prior to and after impact, the time of which is assumed to be negligibly small. Here, T is the kinetic energy in terms of the velocities $(\dot{s}, \dot{\mathbf{y}}^T)$, and the vector $\mathbf{p} = \partial T/\partial\dot{\mathbf{y}}$ is the generalized momentum of the component $\mathbf{y}$ on which no constraint has been imposed during impact.

It is well known[7] that relations (1.4) for the systems of rigid bodies (1.1), (1.2) are only in accord with experiments in the case of a fortunate description of an object which takes account, among other things, of information on the configurations at the instants of the collisions. Otherwise, since, in reality, the impact reactions are simultaneously generated both by a unilateral constraint as well as by a bilateral constraint, part of the kinetic energy may be converted into forms which are not taken into account by the model. The impact will then no longer be absolutely elastic. For example, a bell, which is modelled as a pendulum with a finite number of degrees of freedom, clearly appears as a system with distributed parameters after an impact.[7] In order to remain within the framework of relations (1.4) in the system of rigid bodies with constraints (1.1), we would be obliged to make the model more complex every time depending on the actual form of functions (1.2). In this sense, unilateral constraints cannot always be formally "superposed" on the finite dimensional system (1.1), which has already been assumed. Within the framework of assumptions (1.4) which have been adopted, it is therefore only necessary, henceforth to consider the class of objects in which constrained collisions are not encountered, with the aim of preserving the independent character of relations (1.1) and (1.2). These can be, for example, systems of points masses which are not linked together by constraining bonds and, also, other mechanisms in which the absolute elasticity of the collisions is guaranteed.

Since the dimension of system (1.2) are arbitrary, it is not possible in general to avoid an unsmooth change of coordinates.[5] We shall only use it for the local description of the dynamics in the neighbourhood of the specific hypersurface $f_i(\mathbf{q})=0$. For a complete description of the motions of system (1.1), (1.2), (1.4), we shall use the standard technique of putting together the trajectories from the parts which are separated at the instants of the impacts (the fitting method). The negligibly small durations of the impacts will be "cut from the motion picture tape of events".

Note that they could be left if one is synthetically modelling a viscoelastic medium with an infinite elasticity potential in the domain $f_i(\mathbf{q}) < 0$.[8]

2. Conditions of stabilizability

Conditions of stabilizability, that is, the possibilities of transferring the system from any initial state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ into an ε-neighbourhood of the point $(\mathbf{0}, \mathbf{0})$, were proposed in Refs. 2–4 for systems on semi-invariant sets, that is, which completely contain the phase curves which have started in them. The topic of discussion usually concerns the stabilizability in the whole of phase space, for which the concept of a connected Lyapunov function was introduced.[9]

Definition 1. We shall call a single-valued function $V(\mathbf{y})$ $(\mathbf{y} \in \mathbf{T}^r \times \mathbf{R}^m)$, which is positive-definite on $P \subset \mathbf{T}^r \times \mathbf{R}^m$ in Lyapunov's sense $(\forall \mathbf{y} \in P : V(\mathbf{y}) \geq 0,\ V = 0 \Rightarrow \mathbf{y} = \mathbf{0})$ and continuous together with its partial derivatives, a Lyapunov function.

We will use the notation

$$E_V = \{c\colon c = V(\mathbf{y}), \mathbf{y} \in P\}, \quad H_c(V(\mathbf{y})) = \{\mathbf{y}\colon V(\mathbf{y}) \leq c, c \in E_V\}$$

Definition 2. We shall say that the Lyapunov function $V(\mathbf{y})$ $(\mathbf{y} \in \mathbf{T}^r \times \mathbf{R}^m)$ is connected on $P \subset \mathbf{T}^r \times \mathbf{R}^m$ if each set $H_c(V(\mathbf{y})) \cap P (c \in E_V)$ is connected.

Note that, when unilateral constraints (1.2) are added to system (1.1), the set on which the solutions of the systems are determined will no longer be semi-invariant in the general case since the phase curves can encounter obstructions. Moreover, in systems (1.1), (1.2), certain functions (the potential energies, for example) can become connected Lyapunov functions (CLF) which is solely due to constraints (1.2).

Example 1. For a point m which falls along the z axis on to the horizontal plane $z = 0$, the potential energy $B(z) = mgz$ is a CLF on $P_1 = \{z \in \mathbf{R}^1 : z \geq 0\}$.

We will later formulate one sufficient conditions for a CLF on $P = \{\mathbf{y} \in \mathbf{T}^r \times \mathbf{R}^m : \mathbf{f}(\mathbf{y}) \geq 0\}$, assuming that the set P is connected and that all the hypersurfaces $f_j(\mathbf{y}) = 0 (j = 1,2, \ldots, l)$ are regular.

We will use the notation

$$\partial P = P \backslash P_0, \quad P_0 = \{\mathbf{y} \in \mathbf{T}^r \times \mathbf{R}^m\colon \mathbf{f}(\mathbf{y}) > 0\}$$

We will assume that the surface $\sum_s$, of dimension $s = r + m - k$, can only be an intersection of k of the hypersurfaces $f_i(\mathbf{y}) = 0 (k < l)$ at the points of which k of the vectors $\mathbf{g}_i = \partial f_i / \partial \mathbf{y}$ are linearly independent. The Gram determinant, constructed on these vectors at the point $\mathbf{y} \in \Sigma_s$ will then be positive, that is, det $\{\gamma_{ij}\} > 0$, where $\gamma_{ij} = \mathbf{g}_i^T \mathbf{g}_j (i, j = 1, 2 \ldots, k)$. We will establish a correspondence between the set $N(\mathbf{y}_*)$ of the numbers of all the hypersurfaces $f_i(\mathbf{y}) = 0$ to which it belongs and each point $\mathbf{y}_* \in \partial P$. We will the Gram determinant, constructed on the vectors of the gradients $\mathbf{g}_i(\mathbf{y}_*)(\forall i \in N(\mathbf{y}_*))$, by $\mathbf{T}(\mathbf{y}_*)$. In the case of the auxiliary function $V(\mathbf{y})$ $(\mathbf{y} \in P)$, the vector $\mathbf{b} = \partial V / \partial \mathbf{y}$, calculated at the point $\mathbf{y}_* \in \partial P$, together with the k vectors $\mathbf{g}_i(\mathbf{y}_*)$, generates another $(k + 1)$-th order Gram determinant, which is subsequently denoted by $\Gamma_v(\mathbf{y}_*)$. Local coordinates $\xi_j (j = 1, 2, \ldots, s)$ and the matrix $\partial_2 V_0 / \partial \xi^2$, where $V_0(\xi) = V(\mathbf{y})$, exist in the neighbourhood of the point $\mathbf{y}_* \in \sum_s$.

It is well known (Ref. 10, p. 113) that the smooth function $V(\mathbf{y})$ only has conditional extrema on the smooth surface $f_1(\mathbf{y}) = 0$ at points where $\partial V / \partial \mathbf{y}$ and $\partial f_1 / \partial \mathbf{y}$ are collinear. In general, the conditional extrema of the function $V(\mathbf{y})$ in $\sum_s$ are located at points where the $k + 1$ vectors $\mathbf{b}$, $\mathbf{g}_i$ are linearly dependent, that is, where $\Gamma_v(\mathbf{y}) = 0$. We separate out the subset

$$Q_V(\partial P) = \{\mathbf{y} \in \partial P\colon \Gamma_V(\mathbf{y}) = 0, \mathbf{b}^T \mathbf{g}_i > 0, \forall i \in N(\mathbf{y})\}$$

from the set of conditional critical points of the function $V(\mathbf{y})$ on ∂P and use the notation

$$\psi_V(\varepsilon) = \{\mathbf{y} \in \partial P\colon \Gamma_V(\mathbf{y}) \leq \varepsilon \Gamma(\mathbf{y})\}$$

If $\mathbf{0} \in P_0$, then, by virtue of the smoothness of $V(\mathbf{y})$, we obtain $\mathbf{b}(\mathbf{0}) = \mathbf{0}$. If, however, $\mathbf{0} \in \partial P$, then the point $\mathbf{y} = \mathbf{0}$ cannot be a critical point for $V(\mathbf{y})$ (see Example 1).

Assertion 1. Suppose the Lyapunov function $V(\mathbf{y})$ ($\mathbf{y} \in P$) does not have critical points in P_0 with the exception, perhaps, of the point $\mathbf{y} = \mathbf{0}$ and that an $\varepsilon_0 > 0$ exists such that the set $\Psi_V(\varepsilon)$, for any $\varepsilon < \varepsilon_0$, consists of a finite number of connected compact subsets. If $Q_V(\partial P)\backslash\mathbf{0}$ consists of a finite number of isolated points at which the matrix $\partial^2 V_0/\partial \boldsymbol{\xi}^2$ has a negative eigenvalue, then $V(\mathbf{y})$ is a CLF on P.

Proof. Consider the dynamical system

$$\dot{\mathbf{y}} = -\mathbf{b}(\mathbf{y}) \tag{2.1}$$

on P.

If $\mathbf{0} \in P_0$, then, along each trajectory emerging from $\mathbf{y}(0) \in P_0$ which does not intersect ∂P, we obtain

$$dV/dt = -\mathbf{b}^T\mathbf{b} < 0, \quad \lim_{t \to \infty} V(\mathbf{y}(t)) = 0, \quad \lim_{t \to \infty} \mathbf{y}(t) = \mathbf{0}$$

We will determine the solution of system (2.1) when $f_1(\mathbf{y}(0)) = 0$ using Filippov's rule (Ref. 11, p. 42), assuming that $f_1(\mathbf{y}) = 0$ is the equation of the surface of discontinuity to which the infinitely large velocity vectors $\lambda\mathbf{g}_1$ ($\lambda \to \infty$) are directed from the domain $f_1(\mathbf{y}) < 0$. Depending on the sign of the quantity $e = \mathbf{b}^T\mathbf{g}_1$, the motion either continues in the domain P_0 (if $e \leq 0$) or in a sliding mode along the surface $f_1(\mathbf{y}) = 0$ (when $e > 0$). The sliding velocity

$$\dot{\mathbf{y}} = \alpha\mathbf{b} + (1 - \alpha)\lambda\mathbf{g}_1$$

reduces in the limit when $\lambda \to \infty$ to the form

$$\dot{\mathbf{y}} = -\mathbf{b} + \mu_1\mathbf{g}_1; \quad \mu_1 = e/|\mathbf{g}_1|^2 \tag{2.2}$$

By virtue of equality (2.2), the derivative of the function $V(\mathbf{y})$

$$dV/dt = \mathbf{b}^T\dot{\mathbf{y}} = -\mathbf{b}^T\mathbf{b} + \mu_1\mathbf{b}^T\mathbf{g}_1 \leq 0$$

since $\mathbf{b}^T\mathbf{g}_1 \leq |\mathbf{b}||\mathbf{g}_1|$ (the Cauchy–Bunyakovskii inequality). Here, equality is only attained in the case of codirected vectors $\mathbf{b}$ and $\mathbf{g}_1$, that is, when $\mathbf{y} \in Q_V(\partial P)$.

We now show that, in the case of translation (with the exception of certain trajectories which are receding without disrupting the connectiveness of P), either an ε-neighbourhood of the point $\mathbf{y} = \mathbf{0}$ or a point of descent from the surface $f_1(\mathbf{y}) = 0$ (when $e \leq 0$) or a point of intersection with another surface $f_j(\mathbf{y}) = 0$ ($j \in 1, 2, \ldots, r$) will be reached. We shall at once carry out the arguments for the general case of translation from a point $\mathbf{y}_*$ onto the intersection of several hypersurfaces $f_i(\mathbf{y}) = 0 (\forall i \in N(\mathbf{y}_*))$ when the condition

$$\mathbf{b}^T\mathbf{g}_i > 0 \ (\forall i \in N(\mathbf{y}_*)) \tag{2.3}$$

is satisfied (violation of the inequalities returns the point to the surface of greatest dimension or to P_0). On defining the vector fields in the domains $f_i(\mathbf{y}) < 0$ in the form $\lambda_i\mathbf{g}_i$, $\lambda_i \to \infty (i \in N(\mathbf{y}_*))$, Filippov's rule, we obtain the sliding velocity

$$\dot{\mathbf{y}} = -\mathbf{b} + \Sigma\mu_i\mathbf{g}_i \tag{2.4}$$

where μ_i are the coefficients of the expansion (in the basis $\{\mathbf{g}_i, i \in N(\mathbf{y}_*)\}$) of the projection of the vector $\mathbf{b}$ onto the subspace which is generated by this basis. By virtue of relation (2.4), the time derivative of the function $V(\mathbf{y})$ is as follows:

$$dV/dt = \mathbf{b}^T\dot{\mathbf{y}} = (-\dot{\mathbf{y}} + \Sigma\mu_i\mathbf{g}_i)^T\dot{\mathbf{y}} = -|\dot{\mathbf{y}}|^2 = -\gamma_V; \quad \gamma_V = \Gamma_V(\mathbf{y})/\Gamma(\mathbf{y})$$

When $\gamma_V > \varepsilon$, the rate of decrease in $V(\mathbf{y}(t))$ is finite.

We will now consider in greater detail the case when γ_V, that is, a motion in the set $\Psi_V(\varepsilon)$ which (in the case of a suitable decrease in ε) separates into a finite number of connected compacta, each of which contains no more than a single point from $Q_V(\partial P)$. A $\min\limits_{y \in \psi(\varepsilon)} \gamma_V(\mathbf{y}) = \gamma_\varepsilon$ exists for each compactum. If $\gamma_\varepsilon > 0$, then $dV/dt < -\gamma_\varepsilon$ and the point leaves the set $\psi_V(\varepsilon)$ after a finite time.

If $\gamma_\varepsilon = 0$, we have a neighbourhood of an isolated point $\mathbf{y}_0 \in Q_V(\partial P)$ in which the matrix $\partial^2 V_0/\partial \boldsymbol{\xi}^2 = W$ has a negative eigenvalue. On choosing the local coordinates $\xi_j = \mathbf{y}_{k+j}$ $(j = 1, 2, \ldots, s)$, we obtain

$$\mathbf{y} = (\boldsymbol{\eta}^T, \boldsymbol{\xi}^T)^T, \quad \eta_j = y_j \ (j = 1, 2, \ldots, k)$$

where k is the number of elements in $N(\mathbf{y}_0)$, $s = r + m - k$. The representation $\dot{\boldsymbol{\eta}} = H\dot{\boldsymbol{\xi}}$ is found in view of the linear independence of the vectors $\mathbf{g}_i\ (\mathbf{y}_0)$ $(i \in N(\mathbf{y}_0))$ at the point $\mathbf{y} = \mathbf{y}_0$. Linearization of system (2.1) in the neighbourhood of this point generates the subsystem

$$(I + H^T H)\dot{\boldsymbol{\xi}} = -W(\boldsymbol{\xi} - \mathbf{c})$$

where I is the unit matrix and $\boldsymbol{\xi} = \mathbf{c}$ when $\mathbf{y} = \mathbf{y}_0$. The fact that the matrix W has a negative eigenvalue (according to the condition) guarantees the instability of the solution $\boldsymbol{\xi} = \mathbf{0}$. Elimination of the set Λ of trajectories entering points of the type $\mathbf{y} = \mathbf{y}_0$ from P yields the connected set $P\backslash\Lambda$ in which the remaining trajectories leave the set $\psi_V(\varepsilon)$ after a finite time and reach an ε-neighbourhood of the point $\mathbf{y} = \mathbf{0}$. In view of the monotonicity of $V(\mathbf{y}(t))$ along the solutions of system (2.1) (which form a global focus in $P\backslash\Lambda$), we obtain $H_c(V(\mathbf{y})) \subset H_d(V(\mathbf{y}))$ when $d > c$, that is, the connectedness of all of the sets $H_c(V(\mathbf{y}))$, which it was required to do.

Example 2. The contour lines of the function

$$V(x_1, x_2) = x_2^2 + x_1^2/(1 + x_1^2)$$

are shown in Fig. 1 and the domain $\mathbf{R}^2\backslash P_0$, has been hatched in; the boundary ∂P of this domain asymptotically approaches (from above and downwards to the left) the line $V = c_1$. The Lyapunov function $V(x_1, x_2)$ is connected in $\mathbf{R}^2$ but is not connected in P. The conditions of Assertion 1 are not satisfied in view of the non-compactness of the set $\psi_V(\varepsilon)$.

Example 3. The motions of a mathematical pendulum in the vertical plane (Fig. 2) are hindered by the unilateral constraints $R \le x \le l$, where l is the length of the fibre, R is the radius of the fixed disc with its centre at the suspension, point and φ and x are the polar coordinates of the point mass. In the dimensionless variables $\sigma = (l - R)/l$ $(0 < \sigma < 1)$, $z = (x - l)/l$, $\mathbf{y} = (\varphi, z)^T$, the reduced potential energy $V(\mathrm{y}) = 1 - (z + 1)\cos\varphi$ is a CLF in P:

$$P = \{\mathbf{y} \in \mathbf{S}^1 \times \mathbf{R}^1 : \mathbf{f}(\mathbf{y}) \ge 0\}, \quad \mathbf{f}(\mathbf{y}) = (-z, z + \sigma)$$

Here,

$$\mathbf{b} = \partial V/\partial \mathbf{y} = ((z + 1)\sin\varphi, -\cos\varphi) \ne \mathbf{0} \ \forall \mathbf{y} \in P_0$$

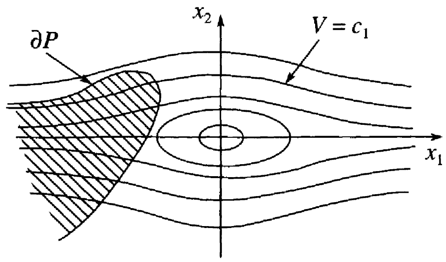

Fig. 1.

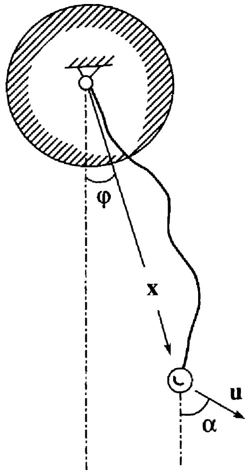

Fig. 2.

The surfaces $f_i(\mathbf{y})=0$ ($i=1, 2$) do not intersect. However, in each of them $\Gamma(\mathbf{y})=\mathbf{g}^T\mathbf{g}=1$ since $\mathbf{g}=\partial f_i/\partial_y=(0, \pm 1)$. The Gram determinant

$$\Gamma_V(\mathbf{y}) = \begin{Vmatrix} \mathbf{b}^T\mathbf{b} & \mathbf{b}^T\mathbf{g} \\ \mathbf{b}^T\mathbf{g} & \mathbf{g}^T\mathbf{g} \end{Vmatrix} = (z_* + 1)^2 2\sin\varphi$$

is only calculated for $z_*=0$ (when $i=1$) or $z_*=-\sigma$ (when $i=2$), and the set $\psi_V(\varepsilon)$ (where $(z_*+1)^2\sin^2\varphi \le \varepsilon \ll 1$) therefore consists of four connected compact neighbourhoods of the points $\delta=0$ and $\varphi=\pi$ for fixed values of z_* ($-\sigma$ or 0). The set $Q_V(\partial P)\backslash\mathbf{0}$ consists of the point $\mathbf{y}_*=(\pi, -\sigma)^T$ at which $\partial^2 V_0/\partial\boldsymbol{\xi}^2=\sigma-1<0$, where $V_0(\boldsymbol{\xi})=V(\varphi, -\sigma)$ when $\varphi=\xi$.

Remark 1. If, in system (1.1) with unilateral constraints (1.2), the potential energy $B(\mathbf{q})$ is a CLF in $P=\{\mathbf{q}\in T^r \times \mathbf{R}:\mathbf{f}(\mathbf{q})\ge \mathbf{0}\}$, then the total energy

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}}/2 + B(\mathbf{q})$$

will be a CLF in TM (1.3).

Actually, in the notation

$$\mathbf{x}_1 = \mathbf{q}, \quad \mathbf{x}_2 = \dot{\mathbf{q}}, \quad T(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2^T A(\mathbf{x}_1)\mathbf{x}_2/2$$

the dynamical system

$$\dot{\mathbf{x}}_2 = -A(\mathbf{x}_1)\mathbf{x}_2, \quad \dot{\mathbf{x}}_1 = \mathbf{0}$$

can be considered in TM. Along its solutions, we have

$$dT/dp = -\mathbf{p}^T\mathbf{p}, \quad \mathbf{p} = A(\mathbf{x}_1)\mathbf{x}_2, \quad \mathbf{x}_1(t)\equiv \mathbf{x}_{10}$$

that is, a monotonic asymptotic decrease in T when $B=\text{const}$. We connect its limit point $(\mathbf{x}_{10}, \mathbf{0})$ to each phase curve. In view of the connectedness of $B(\mathbf{x}_1)$, the continuations of these curves from $(\mathbf{x}_{10}, \mathbf{0})$ to $(\mathbf{0}, \mathbf{0})$ are found (with a monotonic decrease in $B(\mathbf{x}_1)$) in the subspace $\mathbf{x}_2=$ (when $T=0$). As a result, each point $(\mathbf{x}_{10}, \mathbf{x}_{20})\in TM$ will be joined to the point $(\mathbf{0}, \mathbf{0})$ by a line along which the function $E(\mathbf{x}_1, \mathbf{x}_2)$ decreases monotonically. This means, that $E(\mathbf{q}, \dot{\mathbf{q}})$ is an CLF in TM as required.

Remark 2. It can be shown that the points of the conditional extrema of the function $E(\mathbf{q}, \dot{\mathbf{q}})$ in the set $\partial P \times \mathbf{R}^n$ only differ from the analogous points of the function $B(\mathbf{q})$ in ∂P by the formal addition of the components $\dot{\mathbf{q}} = \mathbf{0}$ and, moreover, the nature of the extrema is also repeated (with respect to the number of negative eigenvalues of the second derivatives in the local coordinates).

For prove this, we use the notation

$$V(\mathbf{y}) = E(\mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{y} = (\mathbf{q}, \dot{\mathbf{q}}) \in TM$$

The new vectors $\partial f_i/\partial \mathbf{y}$ will differ from $\partial f_i/\partial \mathbf{q}$ in the addition of n null coordinates and, therefore, $\Gamma(\mathbf{y}) = \Gamma(\mathbf{q})$. It can be shown that

$$\Gamma_V(\mathbf{y}) = \Gamma_E(\mathbf{q}) + \mathbf{p}^T\mathbf{p}\Gamma(\mathbf{q}) \tag{2.5}$$

The Gram determinant $\Gamma_E(\mathbf{q})$ is constructed at the point $\mathbf{q}_* \in \partial P$ for the same vectors $\partial f_i/\partial \mathbf{q}$ $(i \in N(\mathbf{q}_*))$ as in the case of $\Gamma(\mathbf{q})$ but with the addition of the vector $\partial E/\partial \mathbf{q}$ (for an arbitrary $\dot{\mathbf{q}}$) to a number of them. The momentum $\mathbf{p} = A(\mathbf{q}_*)\dot{\mathbf{q}}$ is calculated at the point $\mathbf{y}_* = (\mathbf{q}_*, \dot{\mathbf{q}})$ as is the determinant $\Gamma_V(\mathbf{y})$, which is constructed for the vectors $\partial V/\partial \mathbf{y}$, $\partial f_i/\partial \mathbf{y}$ $(i \in N(\mathbf{y}_*))$.

If $\Gamma_V(\mathbf{y}) = 0$, then, by virtue of the condition $\Gamma_V(\mathbf{q}) \neq 0$, we obtain $\mathbf{p} = \mathbf{0}$ from equality (2.5) (and this means that $\dot{\mathbf{q}} = \mathbf{0}$, $\partial E/\partial \mathbf{q} = \partial P \times \mathbf{R}^n$ and $\Gamma_E(\mathbf{q}) = 0$, that is $\Gamma_B(\mathbf{q}) = 0$). Consequently, the conditional critical points of the function $V(\mathbf{y})$ in $\partial P \times \mathbf{R}^n$ will be the same as in the case of the function $B(\mathbf{q})$ in ∂P (with the addition of n null coordinates in view of the equality $\dot{\mathbf{q}} = 0$). The type of these points are also identical (with respect to the number of negative eigenvalues of the second derivatives in the local coordinates $\mathbf{y}_1 = (\boldsymbol{\zeta}, \dot{\mathbf{q}})$ when $\dot{\mathbf{q}} = \mathbf{0}$, $\mathbf{q}_* \in \partial P$), since

$$\partial^2 V_0/\partial \mathbf{y}_1^2 = \operatorname{diag}(\partial^2 B_0/\partial \xi^2, A(\mathbf{q}_*))$$

as required.

Assertion 2. Suppose that, in system (1.1) with the unilateral constraints (1.2) (and conditions (1.4) in the case of collisions), the potential $B(\mathbf{q})$ is a CLF in P and all the sets $H_c(B(\mathbf{q}))$ $(c \in E_B)$ are compact. Then, if, during free motion, the system does not admit of the particular solution $\dot{q}_j \equiv 0$ (excluding equilibrium positions), then it is stabilizable with respect to an input u_j, $(j \in 1, 2, \ldots, n)$ in the set $TM \backslash \Omega(u_j)$.

Proof. According to Remark 1, the total energy $E(\mathbf{q}, \dot{\mathbf{q}})$ is a CLF in TM. The sets $H_c(E(\mathbf{q}, \dot{\mathbf{q}}))$ are compact in TM by virtue of the boundedness of $|\dot{\mathrm{q}}|$ when $\dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}}/2 \leq c$. We now choose a smooth function $u_j(\mathbf{q}, \dot{\mathbf{q}})$ such that

$$|u_j(\mathbf{q}, \dot{\mathbf{q}})| \leq a_j, \quad \operatorname{sign} u_j = -\operatorname{sign} \dot{q}_j$$

When $\mathbf{q}(0) \in P_0$, by virtue of system (1.1) we obtain

$$dE/dt = -u_j \dot{q}_j \leq 0 \tag{2.6}$$

that is, $E(t) \leq E(0) = c$ and, in the compactum $H_c(E(\mathbf{q}, \dot{\mathbf{q}}))$, the solutions of system (1.1) are continuable for any time interval while $\mathbf{q}(t) \in P_0$. When the set ∂P is reached (for example, the surface $f_1(\mathbf{q}) = 0$ when $t = t_1$), the subsequent motion depends on the quantity $\dot{s}_- = \dot{s}(t_1 - 0)$, where $s = \mathrm{f}_1(\mathbf{q})$. If $s_- \neq 0$, an impact occurs after which, by virtue of relations (1.4), $T(\mathbf{q}, \dot{\mathbf{q}})$, $B(\mathbf{q})$, and this also means $E(\mathbf{q}, \dot{\mathbf{q}})$, turn out to be continuous functions. We note that, in the case of a multiple impact (when $f_j(\mathbf{q}) = 0, f_i(\mathbf{q}) = 0$, $i \neq j$ simultaneously), the three above-mentioned functions have no discontinuities, that is, condition (2.6) is not violated. If $\dot{s}_- = 0$, the motion will occur by virtue of the equations with a Lagrange multiplier

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{u} + \lambda \frac{\partial f_1}{\partial \mathbf{q}}, \quad f_1(\mathbf{q}) = 0 \tag{2.7}$$

while the scalar quantity which expresses the reaction force of the constraint is positive.

At the same time,

$$\frac{dE}{dt} = \left(\mathbf{u} + \lambda\frac{\partial f_1}{\partial \mathbf{q}}\right)^T \dot{\mathbf{q}}, \quad \left(\lambda\frac{\partial f_1}{\partial \mathbf{q}}\right)^T \dot{\mathbf{q}} = 0$$

and, in the general case of motion along the intersection of several hypersurfaces $f_i(\mathbf{q}) = 0$, we obtain $(\sum \lambda_i \partial f_i/\partial \mathbf{q})^T \dot{\mathbf{q}} = 0$, and condition (2.6) is therefore also not violated.

If $\Omega = \Omega(u_j(\mathbf{q}, \dot{\mathbf{q}}))$, then $\dim\Omega < 2n$ and the domain $TM\backslash\Omega(u_j)$ will be open and everywhere dense in TM (in view of the non-compactness of TM and the continuity of $E(\mathbf{q}, \dot{\mathbf{q}})$). There are no complete trajectories of the system for the set $dE/dt = 0$ as $u_j = 0$ only when $\dot{q}_j = 0$ and the particular solution $\dot{q}_j \equiv 0$ (when $\mathbf{u} \equiv \mathbf{0}$) is missing according to the condition. As a result, the conditions of the Barbashin–Krasovskii theorem, modified for $\mathbf{T}^r \times \mathbf{R}^{2n-r}$,[9] are satisfied, that is, the asymptotic stability of the zero solution of system (1.1) with feedback $u_j(\mathbf{q}, \dot{\mathbf{q}})$ holds for $TM\backslash\Omega(u_j)$. Hence, system (1.1) is stabilized with respect to an input u_j ($j \in 1, 2, \ldots, n$) on the set $TM\backslash\Omega(u_j)$ which it was required to demonstrate.

3. Conditions for global controllability

In order to establish the controllability properties of natural Lagrangian systems, use has been made of their stabilizability to an equilibrium state (or to a certain steady motion), local controllability in the neighbourhood of such a system and the symmetry of the equations of motion with respect to time reversal

$$\dot{\mathbf{q}} \to -\dot{\mathbf{q}}, \quad t \to -t \tag{3.1}$$

when the possibility of motion "in reverse time"

$$\mathbf{u}(t): (\mathbf{q}_1, \dot{\mathbf{q}}_1) \to (\mathbf{q}_2, \dot{\mathbf{q}}_2), \quad t \in [0, \tau]$$

follows from the existence of a controllable trajectory

$$\mathbf{u}(\tau - t): (\mathbf{q}_2, -\dot{\mathbf{q}}_2) \to (\mathbf{q}_1, -\dot{\mathbf{q}}_1)$$

The use of appropriate "attainable curves" will be a further generalization of this approach. This enables as, in particular, to give up the requirements of local null-controllability,[1] which is absent in the general case when $0 \in \partial P$. We shall rely on the time reversibility of the trajectories of system (1.1), (1.2), (1.4), which are joined by fitting procedures. In the case of such an object, he existence of null-controllability also guarantees global controllability, that is, the possibility of transfer from any state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ to any required state $(\mathbf{q}_f, \dot{\mathbf{q}}_f)$ after a finite time under the permissible control $\mathbf{u}(t)$. In fact, the feasibility of the transition $(\mathbf{q}_f, -\dot{\mathbf{q}}_f) \to (\mathbf{0}, \mathbf{0})$ implies (by virtue of symmetry condition (3.1)) the possibility of the transition $(\mathbf{0}, \mathbf{0}) \to (\mathrm{q}_f, \dot{\mathrm{q}}_f)$, and the motion $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (0, 0) \to (\mathbf{q}_f, \dot{\mathbf{q}}_f)$ therefore exists for any $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ and $(\mathbf{q}_f, \dot{\mathbf{q}}_f)$.

Definition 3. We shall call a closed curve corresponding to a periodic motion (with a finite period) and containing a globally accessible point, that is, a point $(\mathbf{q}_*, \dot{\mathbf{q}}_*) \in TM$ into which the system can be transferred by an admissible control after a finite time from any other point of phase space, a globally accessible curve in the phase space of system (1.1), (1.2), (1.4).

Assertion 3. The necessary and sufficient condition for the global controllability of the natural Lagrangian system (1.1), (1.2), (1.4) is the existence of a globally accessible curve containing both a certain point $(\mathbf{q}_*, \dot{\mathbf{q}}_*)$ and the point $(\mathbf{q}_*, -\dot{\mathbf{q}}_*)$ which is symmetric to it.

Proof. The necessity is verified, for example, by the choice of the accessible periodic motion $\mathbf{q} \equiv 0$, $\dot{\mathbf{q}} \equiv 0$ since its null-controllability follows from the global controllability of the system.

The sufficiency follows from the existence (according to Definition 3) of the motion $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (\mathbf{q}_*, \dot{\mathbf{q}}_*)$ from any initial point $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$. If $(\mathbf{0}, \mathbf{0})$ is taken as such a point, then, by virtue of condition (3.1)), the symmetric transition $(\mathbf{q}_*, -\dot{\mathbf{q}}_*)(\mathbf{0}, \mathbf{0})$ is found together with the transition $(\mathbf{0}, \mathbf{0})(\mathbf{q}_*, -\dot{\mathbf{q}}_*)$. As a result, the stepwise motion

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0) \to (\mathbf{q}_*, \dot{\mathbf{q}}_*) \to (\mathbf{q}_*, -\dot{\mathbf{q}}_*) \to (\mathbf{0}, \mathbf{0})$$

will be guaranteed, which means the null-controllability of system (1.1), (1.2), (1.4) and, consequently, also its global controllability, as it was required to prove.

Corollary. *For the global controllability of the natural Lagrangian system* (1.1), (1.2), (1.4), *it is necessary and sufficient that a globally accessible curve exist containing a point of the form* $(\mathbf{q}_*, \mathbf{0})$.

Note that, in the analysis of controllability, two cases are qualitatively distinguished (depending on the location of the rest point $\mathbf{q} = \mathbf{0}$ with respect to the set (1.2)): 1) the case $\mathbf{f}(\mathbf{0}) > \mathbf{Q}$ when unilateral constraints have not been applied in the equilibrium state; then, for global controllability, stabilizability and local controllability are sufficient, and it is sufficient[1] to reveal the latter in the linear approximation on which the existence of constraints (1.2) has no effect; 2) the case when $f_1(\mathbf{0}) = 0$ when equilibrium is attained due to a unilateral constraint (which we assume to be the first); here, the linear approximation cannot be used in the neighbourhood of the state of rest in view of the discontinuity of the right-hand side of the differential equation.

We will now consider the second case in greater detail assuming (in accordance with Remark 1) that it is impossible for the point $\mathbf{q} = \mathbf{0}$ to belong at once to the intersection of two or more hyperplanes $f_i(\mathbf{q}) = 0$ (otherwise we would be able to rely on the special change of variables in Ref. 5 which is used later). For the proof of global controllability, we shall seek globally accessible curves and show, for example, that these curves are explicitly encountered in systems with two degrees of freedom.

4. Globally controllable systems with two degrees of freedom

We now consider system (1.1), (1.2), (1.4) when $n = 2$ under just a single control u_1. We will assume that this control allowed the system, during the stabilization process, to be brought from unstable equilibrium states (and from the set Ω), that is, stabilizability in TM was ensured as a result. We will also assume that, in the ε-neighbourhood of the point $(\mathbf{0}, \mathbf{0}) \in TM$, which has already been reached, it is possible by means of an appropriate change of coordinates ($s = f_1(\mathbf{q})$, for example) to transfer to the configuration $\mathbf{q} = (s, y)^T$ for which the first inequality of (1.2) has the form $s \geq 0$ and the remaining inequalities do not constrain the motions, that is, for small $|\mathbf{q}|$, $|\dot{\mathbf{q}}|$, only impacts on the surface $s = 0$ prove to be possible. Finally, we will assume that the canonical structure[12] of the kinetic energy

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\lambda\dot{s}^2 + \frac{1}{2}h(s, y)\dot{y}^2, \quad \lambda > 0, \quad h(s, y) > 0$$

is attained by a suitable choice of the y coordinate.

Denoting the potential energy by $B(s, y)$, we obtain the equations of motion

$$\lambda\ddot{s} - \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial s} + \frac{\partial B}{\partial s} = ru, \quad s \geq 0; \quad h\ddot{y} + \dot{s}\dot{y}\frac{\partial h}{\partial s} + \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial y} + \frac{\partial B}{\partial y} = \rho u, \quad |u| \leq a$$

The coefficients $r = r(s, y)$, $\rho = \rho(s, y)$ accompanying the control appeared as a consequence of the transition from the old configuration to the new one.

In the ε-neighbourhood of the equilibrium state being considered

$$s = 0, \quad \dot{s} = 0, \quad y = 0, \quad \dot{y} = 0 \tag{4.1}$$

we obtain the equations without kinematic constraints

$$\begin{aligned} \lambda\ddot{x} &= \left(ru + \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial|x|} - \frac{\partial B}{\partial|x|}\right)\operatorname{sign}x \\ h\ddot{y} + \frac{\partial h}{\partial|x|}\dot{x}\dot{y}\operatorname{sign}x + \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial y} + \frac{\partial B}{\partial y} &= \rho u \end{aligned} \tag{4.2}$$

by means of the unsmooth substitution[5] $s = |x|$. At the same time the equality $\partial B/\partial y = 0$ is satisfied in the state (4.1) since no constraint is imposed on y.

Next, we use the notation

$$\left.\frac{\partial B}{\partial |x|}\right|_{y=0} = \delta(x) \tag{4.3}$$

and, depending on the value of $\delta(0)$, we shall speak of an equilibrium with a positive reaction ($\delta(0)>0$) or a null reaction ($\delta(0)=0$) of the unilateral constraint.

We initially consider the first case. If the magnitude of $\delta(0)$ is positive and finite, then, in the domain of the motions being considered, it exceeds both the values of $|\mathbf{q}|$, $|\dot{\mathbf{q}}||\partial B/\partial y|$ as well as the control resource a, which is taken (according to the condition) to be as small as desired.

Remark 3. The existence of a value $\varepsilon_0>0$ such that, for any $\varepsilon>\varepsilon_0$, $\rho(\mathbf{q})\neq 0$ ($\forall \mathbf{q}\in H_\varepsilon(B(\mathbf{q}))$ is satisfied is a necessary condition for the controllability of system (4.2) when $0<|ra|<\delta(0)$. Actually, the state (4.1) will otherwise be an invariant set which it is impossible to leave in the case of any admissible control whatsoever and, since the expressions in brackets is negative in the first equation of (4.2), a "translation" state $x\equiv 0$ arises.

We next assume that the condition of Remark 3 is satisfied and calculate the values

$$\min_{\mathbf{q}}\rho = \rho_0, \quad \max_{\mathbf{q}} h = h_0$$

in the compactum $H_\varepsilon(B(\mathbf{q}))$.

In the ε-neighbourhood of the point $(\mathbf{0},\mathbf{0})\in TM$, we represent the equation in the form

$$u = u_1+u_2; \quad u_1 = \frac{wh}{\rho}, \quad u_2 = \frac{1}{\rho}\left(\frac{\partial h}{\partial |x|}\dot{x}\dot{y}\operatorname{sign}x + \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial y} + \frac{\partial B}{\partial y}\right) \tag{4.4}$$

and, in view of the smallness of $|\mathbf{q}|$, $|\dot{\mathbf{q}}|$, the inequalities

$$|u_1|\le a/2, \quad |u_2|\le a/2$$

will be ensured if $w(t)$ is chosen from the condition $|w|\le a_1$, $a_1=a\rho_0/(2h_0)$.

Substituting expression (4.4) into system (4.2), we obtain

$$\begin{gathered}\ddot{y} = w, \quad |w|\le a_1, \quad \lambda\ddot{x} + \frac{\partial B}{\partial |x|}\operatorname{sign}x = G\operatorname{sign}x\\ G = \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial |x|} + \frac{r}{\rho}\left(wh + \frac{\partial h}{\partial |x|}\dot{x}\dot{y}\operatorname{sign}x + \frac{1}{2}\dot{y}^2\frac{\partial h}{\partial y} + \frac{\partial B}{\partial y}\right)\end{gathered} \tag{4.5}$$

When the conditions

$$y = 0, \quad \dot{y} = 0, \quad w = 0 \tag{4.6}$$

are satisfied, we have $G=0$ and, by choosing an admissible control $w(t)$, it is possible to transfer system (4.5) to the state (4.6) for which the second equation of (4.5) is integrated in the form

$$H(x,\dot{x}) = \lambda\dot{x}^2/2 + B(|x|,0)\operatorname{sign}x = c_1 \tag{4.7}$$

When $\delta(0)>0$, the integral (4.7) describes natural oscillations about the constraint and, in the phase plane $(x,\dot{x})$, it is depicted by a closed curve which is symmetrical about the centre. The curve (4.6), (4.7) proves to be globally accessible if the value c_1 which is specified in advance is successfully reached. By virtue of Eq. (5.5), the time derivative of the function $H(x,\dot{x})$ has the form

$$\frac{dH}{dt} = \dot{x}J\operatorname{sign}x, \quad J = G + \left.\frac{\partial B}{\partial |x|}\right|_{y=0} - \frac{\partial B}{\partial |x|} \tag{4.8}$$

In order to obtain the required value $H=c_1$ (in the ε-neighbourhood of the point $(\mathbf{0},\mathbf{0})\in TM$ in the case of the state (4.6), (4.7)), it is possible initially to transfer the system (by means of a stabilizing control) into the domain of sufficiently small values of $H < c_1$, at the same time reaching the state (4.6) (with the choice of $w(t)$) and then to use the following stepwise procedure of monotonically increasing H. In the next stage, $T = 0$ is set at the instant of time when $x = 0$, $\dot{x} = \sqrt{2H}$. In the interval $t \in [0, \tau]$ (where τ is the time of the motion in the domain $x \geq 0$, $\dot{x} \geq 0$), we specify the preset control

$$w(t) = a_2 w_0(t); \quad |a_2| \leq a_1,$$

$$w_0(t) = \begin{cases} 1 \text{ при } t \in [0, \tau/4[\cup[3\tau/4, \tau] \text{ или при } t > \tau \\ -1 \text{ при } t \in [\tau/4, 3\tau/4[\end{cases} \tag{4.9}$$

Condition (4.6) is satisfied at the beginning and end of such a manoeuvre and the resulting increment in the function H will depend on a_2 in the form

$$\Delta H(a_2) = \int_0^\tau \dot{x} J \operatorname{sign} x dt \tag{4.10}$$

The principal part of the increment (4.10) can be obtained using an asymptotic expansion, introducing the small parameter $\nu = 2H(0)$ and recalculating the scales of the variables and the time using the formulae

$$x = \nu x', \quad y = \nu y', \quad H = \nu H', \quad a_1 = \nu a_1', \quad t = \sqrt{\nu} t' \tag{4.11}$$

Then, omitting the prime, we represent

$$x = x_0 + \nu x_1 + 0(\nu^2), \quad J = J_0 + \nu J_1 + 0(\nu^2), \quad H = H_0 + \nu H_1 + 0(\nu^2) \tag{4.12}$$

and we write the zeroth approximation for Eqs. (4.5) and (4.8) as follows:

$$\lambda \ddot{x}_0 + \delta(x_0) \operatorname{sign} x_0 = 0, \quad dH_0/dt = 0 \tag{4.13}$$

On finding specific functions $x_0 \geq 0, \dot{x}_0 \geq 0$ in the segment $t \in [0, t_1] (t_1 = \tau/\sqrt{2H_0})$ from the first equation of (4.13), we estimate (using the known $w_0(t)$, $y(t)$, $\dot{y}(t)$) the value of the integral (4.10) in the form

$$\Delta H(a_2) \approx \nu H_1(\tau), \quad H_1 = \int_0^t \dot{x}_0(\xi) J_0(\xi) d\xi \tag{4.14}$$

Since the function $H_1(t)$ is calculated in quadratures, the quantity

$$\Delta = \max_{a_2} H_1(t_1) \tag{4.15}$$

is determined explicitly. As H increases, the parameter ν increases and, therefore, if the value of Δ is positive and finite, then, by repeating the strategy (4.9) many times, it is possible to attain the specified (sufficiently small) value of $H = c_1$ after a finite number of steps. In other words, in the phase space TM the curve (4.6), (4.7) proves to be globally accessible and contains the point $\dot{\mathbf{q}} = \mathbf{0}$. Then, according to the corollary from Assertion 3, the following sufficient condition of controllability will hold.

Remark 4. Suppose the system (1.1), (1.2), (1.4) with two degrees of freedom is stabilized and, in the equilibrium state (4.1), the reaction of the unilateral constraint (4.3) is positive. If, for the equations in the form of (4.5), the quantity Δ, obtained in the case of the motion from the state (4.6) with the control (4.9), is positive and finite, then the system is globally controllable.

Example 4. We will now consider the mathematical pendulum with the kinematic constraints from Example 3 (Fig. 2) as the controlled system. Suppose the control force u ($|u| \leq a$, a is a given quantity) is applied to the pendulum at an acute angle α to the vertical (the model of relative motion for a varying acceleration of the suspension point parallel to an inclined line). In the notation $\mathbf{q} = (\varphi, z)^T$, the system has the potential $B(\mathbf{q}) = 1 - (z + 1)\cos\varphi$ which is a CLF (see Example 3) and a total energy

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \dot{z}^2/2 + (1+z)^2\dot{\varphi}^2/2 + B(\mathbf{q})$$

Moreover, all of the sets $H_c(B(\mathbf{q}))$ $(c \in E)$ are compact in view of the boundedness of φ and z.

By virtue of the equations of motion

$$\begin{aligned} &\ddot{z} - (1+z)\dot{\varphi}^2 - \cos\varphi = u\cos(\alpha - \varphi) \\ &(1+z)\ddot{\varphi} + 2\dot{z}\dot{\varphi} + \sin\varphi = u\sin(\alpha-\varphi), \quad -\sigma \le z \le 0 \end{aligned} \tag{4.16}$$

the time derivative of $E(\mathbf{q}, \dot{\mathbf{q}})$ will be

$$dE/dt = u[\dot{z}\cos(\alpha-\varphi) + (1+z)\dot{\varphi}\sin(\alpha-\varphi)]$$

When $u \equiv 0$, the expression in square brackets is only equal to zero for equilibrium states (since free motion of the point along the line

$$(1+z)\cos(\alpha-\varphi) = \text{const}$$

inclined at an angle $\alpha \neq 90°$ to the horizontal is impossible and, in the case of motion along the constraint when $\dot{z} = 0$, we have $\dot{\varphi} \neq 0$). The set $\Omega(u)$ consists of the trajectories entering the point

$$\varphi = \pi, \quad \dot{\varphi} = 0, \quad z = -\sigma, \quad \dot{z} = 0$$

from which the object can evolve by means of an adrmissible control while on a lower energy level. By virtue of Assertion 2, the system is therefore stabilized in TM with respect to an input u, that is, it can be transferred from any state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ to the ε-neighbourhood of the point $(\mathbf{0}, \mathbf{0})$. In this neighbourhood, a continuous description of the motion[5] is achieved by the substitution $z = |x|$, since the kinetic energy already has a canonical structure. Since $\partial B/\partial |x|$, then $\delta(0) = 1$, that is, the reaction of the constraint is positive in the equilibrium state. The vertical motion of the pendulum with impacts at the instants when the fibre is taut and the conservation law

$$H(x, \dot{x}) = \dot{x}^2/2 + |x| = c_1$$

corresponds to condition (4.6).

In the phase space TM, this will be a globally accessible curve since, after a finite number of cycles of control (4.9), the preassigned value of c_1 can be attained.

It can be shown that, in the asymptotic expansion procedure (4.12), the estimate $\Delta H(a_2)$ in the first approximation is characterized by the quantity (4.15) in the form

$$\Delta = \max_{a_2}\left|\frac{a_2}{32}\left(\frac{3}{2}\operatorname{ctg}\alpha - a_2\right)\right| \tag{4.17}$$

Actually, here, Eq. (4.5) acquires the form

$$\ddot{y} = w, \quad |w| \le a_1, \quad \ddot{x} + \cos\varphi\,\text{sign}\,x = G\,\text{sign}\,x$$

$$G = -(1-|x|)\dot{\varphi}^2 - \operatorname{ctg}(\alpha-\varphi)[(1-|x|)w - 2\dot{x}\dot{\varphi}\,\text{sign}\,x + \sin\varphi]$$

After recalculation of the scales (4.11) and the asymptotic expansion (4.12), we obtain the solution

$$x_0 = 1 - t, \quad x_0 = t - t^2/2 \ (t \in [0, 1])$$

for the zeroth approximation of (4.13) in the positive quadrant of the phase plane $(x, \dot{x})$.

For the same time interval, the integrable component

$$\dot{x}_0[(1-x_0)a_2w_0 - 2\dot{x}_0\dot{\varphi} + \varphi)] = dN/dt$$

is separated out in the equation of the first order approximation

$$dH_1/dt = -\dot{x}_0[\dot{\varphi}^2 + \operatorname{ctg}\alpha(a_2w_0 - 2\dot{x}_0\dot{\varphi} + \varphi)]$$

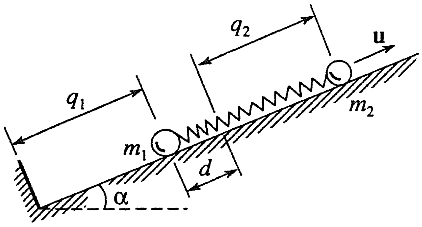

Fig. 3.

where $N=(1-x_0)(\varphi+\dot{x}_0\dot{\varphi})-\dot{x}_0^2\dot{\varphi}$ (when account is taken of the equalities $\ddot{\varphi}=a_2w_0$, $\ddot{x}_0=-1$). It therefore follows from the expression for H_1 (4.14) that

$$H_1(t) = -I_1(t)-a_2\operatorname{ctg}\alpha[I_2(t)-N(t)]$$

$$I_1(t) = \int_0^t \dot{x}_0(\xi)\varphi^2(\xi)d\xi,\quad I_2(t) = \int_0^t \dot{x}_0(\xi)w_0(\xi)x_0(\xi)d\xi$$

Since $\varphi(1)=\dot{\varphi}(1)=0$, we have $N(1)=0$. Moreover, we obtain by direct integration

$$I_1(t) = a_2^2/96,\quad I_2(t) = -3/64$$

whence, in the notation (4.15), we have equality (4.17).

By virtue of Remark 4, the system (Fig. 2) is globally controllable, that is, it can be transferred after a finite time by a bounded control $|u|\le a$ (where the magnitude of a is specified and can be as small as desired) from any initial state to any required state $(x_f, \dot{x}_f, \varphi_f, \dot{\varphi}_f)$ when there are unilateral constraints.

Example 5. The system (Fig. 3) consists of two bodies of negligibly small dimensions with masses m_1 and m_2. Fastened by a spring, they move translationally and rectilinearly along an inclined plane without friction. The coordinate q_1 of the first body is measured from the fixed obstacle. The length of the spring in the unstressed state is equal to l and its stiffness c satisfies the condition

$$d = l-(m_2g\sin\alpha)/c>0$$

that is, the bodies do not come into contact in the equilibrium state (when $q_1=0$) The coordinate q_2 of the body with mass m_2 is measured such that q_2+d is the distance between the bodies. In view of the constraints $q_1\ge 0$, $q_2+d\ge 0$, impacts (which are assumed to be absolutely elastic) between the bodies and of the body m_1 on the obstacle are possible. An external control force u is applied to the body with mass m_2, $|u|\le a$ (where a is a specified quantity and can be as small as be desired). Introducing the dimensionless variables and the dimensionless time

$$\mu = \frac{m_2}{m_1},\quad q_i' = \frac{q_i}{l}(i=1,2),\quad u' = \frac{u}{m_1g},\quad a' = \frac{a}{m_1g},\quad d' = \frac{d}{l},\quad c' = \frac{cl}{m_1g},\quad t' = t\sqrt{\frac{g}{l}}$$

we obtain, omitting the prime, the reduced Lagrang function

$$L(\mathbf{q},\dot{\mathbf{q}}) = \dot{q}_1^2/2+\mu(\dot{q}_1+\dot{q}_2)^2/2-B(\mathbf{q})$$

in the coordinates $\mathbf{q}=(q_1, q_2)^T$, where the potential energy

$$B(\mathbf{q}) = cq_2^2/2+(1+\mu)q_1\sin\alpha$$

is a Lyapunov function in $p=\{\mathbf{q}\in\mathbf{R}^2{:}q_1\ge 0,\ q^2+d\ge 0\}$. The sets $H_c(B(\mathbf{q}))$ in $\mathbf{R}^2$ (which are out off from the right angle P by parabolae with a common axis $q_2=0$) are compact and the function $B(\mathbf{q})$ is a CLF in P. By virtue of the equations of motion

$$(1+\mu)\ddot{q}_1+\mu\ddot{q}_2+(1+\mu)\sin\alpha = u,\quad \mu(\ddot{q}_1+\ddot{q}_2)+cq_2 = u \tag{4.18}$$

with the constraints

$$q_1 \geq 0, \quad q_2 + d \geq 0 \tag{4.19}$$

the time derivative of the total energy $E(\mathbf{q}, \dot{\mathbf{q}})$ will be $dE/dt = u(\dot{q}_1 + \dot{q}_2)$ and, moreover, in the free motion ($u \equiv 0$), the particular solution $\dot{q}_1 + \dot{q}_2 \equiv 0$ is only possible in the equilibrium state $\mathbf{q} \equiv \mathbf{0}$. According to Assertion 2, system (4.18), (4.19) can be successfully transferred after a finite time by means of an admissible control from any initial state $(\mathbf{q}_0, \dot{\mathbf{q}})$ to the ε-neighbourhood of the point $(\mathbf{0}, \mathbf{0})$. In this neighbourhood, we next make the substitution[5]

$$q_1 = |x|, \quad y = \mu(q_1 + q_2)$$

where $\dot{y} = \partial T/\partial \dot{q}_2$ (the momentum of the "non-collision" coordinate) such that the function $\dot{y}(t)$ is continuous during collisions of the body of mass m_1 with the obstacle, but collisions between the bodies with masses m_1 and m_2 are now impossible in view of the smallness of x and y. In the equilibrium state ($x = 0$, $y = 0$), the reaction of unilateral constraint (4.3)

$$\delta(0) = (1 + \mu)\sin\alpha > 0$$

In the domain of motions being considered, Eq. (4.5) take the form

$$\ddot{y} = w, \quad |w| \leq a_1, \quad \ddot{x} + cx[(1 + \mu)\sin\alpha - \omega^2 y]\operatorname{sign}x = 0, \quad \omega = \sqrt{c/\mu}$$

since, here, $r = 0$, $\rho = 1$, $h = 1/\mu$, $G = 0$. With condition (4.6), the integral (4.7)

$$H(x, \dot{x}) = \dot{x}^2/2 + cx^2/2 + (1 + \mu)|x|\sin\alpha = c_1$$

describes the process of the periodic collisions of the body of mass m_1 with the obstacle during which the body of mass m_2 is maintained at rest by the component u_2 of the control. A globally accessible curve corresponds to this motion in the phase space TM since, after several cycles of control (4.9), the preassigned value of c_1 can be attained. Actually, by solving Eq. (4.13) in the domain $x_0 \geq 0$, $\dot{x}_0 \geq 0$, we obtain estimate (4.15) in the form

$$\Delta = \max_{a_2} \int_0^\tau \dot{x}_0(\xi) J_0(\xi) d\xi$$

where $J_0(t) = \omega^2 y(t) > 0$ (when $t \in]0, \tau]$) and, therefore, $\Delta > 0$.

According to Remark 4, system (4.18), (4.19) (Fig. 3) is globally controllable in the case of a control resource a, which can be as small as desired.

Note that local null-controllability is missing in Examples 4 and 5 since, when $x(0) = \dot{x}(0) = 0$ and in the case of small y, a sliding process $x \equiv 0$ arises, which it is impossible to leave without significant "swinging" of the system.

5. A unilateral constraint with a null reaction at equilibrium

We will now consider the case when $\delta(0) = 0$ which, by virtue of (4.3), corresponds to a null reaction of the unilateral constraint at equilibrium. Again, system (1.1), (1.2), (1.4) with two degrees of freedom is the topic of discussion which, as a result of the stabilization, is brought into the ε-neighbourhood of the point $(\mathbf{0}, \mathbf{0}) \in TM$. We will investigate the equations of motion (4.2) in this neighbourhood, assuming in the particular case that $r > 0$. This assumption narrows down the range of systems being discussed but it enables us to indicate explicitly the globally accessible curve for them. As before, $\rho \neq 0$.

In view of the smallness of the quantities $|\mathbf{q}|$, $|\dot{\mathbf{q}}|$ in the domain of motion being considered, the factor in front of sign$x$ in the first equation of (4.2) will have a sign which is determined by the term ru. Within the framework of the preassigned constraint $|u| \leq a$, the control can be selected in the form

$$u = u_3 + u_4, \quad u_4 = \frac{1}{r}\left(\frac{\partial B}{\partial |x|} - \frac{1}{2}\dot{y}^2 \frac{\partial h}{\partial |x|}\right), \quad |u_3| \leq \frac{a}{2} \tag{5.1}$$

thereby reducing the first equation of (4.2) to the form

$$\lambda \ddot{x} = ru_3 \operatorname{sign} x \tag{5.2}$$

We initially transfer system (5.2) the state

$$x = 0, \quad \dot{x} = 0 \tag{5.3}$$

using a suitable control $u_3 = w\operatorname{sign}x/(\lambda r)$, choosing the function $w(t)$ from the accessible domain in accordance with the equation $\ddot{x} = w$. The motion with the initial conditions (5.3) can then be continued into the sliding process $x(t) \equiv 0$ if the inequality $ru_3 < 0$ is ensured, for example, by means of the control

$$u_3 = (u_5 - \chi)/r, \quad \chi > a_5 \geq |u_5(t)| \tag{5.4}$$

The numbers χ and a_5 are chosen from the third condition of (5.1) and exceed the magnitudes of $|\mathbf{q}|$, $|\dot{\mathbf{q}}|$. The specific value of χ and the function $u_5(t)$ will be determined from the objectives of the control by the state y, $\dot{y}$ when $x(t) \equiv 0$. Substituting relations (5.1), (5.3) and (5.4) into the second equation of (4.2), we obtain

$$h(0, y)\ddot{y} + \frac{1}{2}\dot{y}^2[F_1(y) + F_2(y)F_3(y)] + F_2(y)[F_4(y) + \chi] = F_2(y)u_5 \tag{5.5}$$

where all of the functions

$$F_1(y) = \frac{\partial h}{\partial y}, \quad F_2(y) = \frac{\rho}{r}, \quad F_3(y) = \frac{\partial h}{\partial |x|}, \quad F_4(y) = \frac{r}{\rho}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial |x|} \tag{5.6}$$

are calculated when $x \equiv 0$.

Remark 5. It can be shown that, when the conditions

$$F_3(y) \equiv 0, \quad F_2(y)dF_4/dy > 0 \tag{5.7}$$

are satisfied, system (5.5), in the domain of motions being considered, can be transferred after a finite time to a certain equilibrium state

$$y = y_*, \quad \dot{y} = 0 \tag{5.8}$$

where the value of y_* is determined by the specification of the sufficiently small quantity χ.

In fact, the function $F_4(\dot{y})$, defined by the last equality of (5.6), satisfies the condition $F_4(0) = 0$ by virtue of the assumptions which have been made concerning the null values of the reaction of the constraint in the equilibrium state. In view of condition (5.7), the function $F_4(y)$ is monotonic, and this means that it has an inverse which takes, for example, the value

$$F_4^{-1}(-\chi) = y_* \tag{5.9}$$

in the case of the magnitude of χ which is selected later. When $u_5 = 0$, system (5.5) then has an equilibrium state (5.8) and is locally controllable in its neighbourhood since the linear approximation

$$g_1\ddot{\eta} + g_2\eta = u_5, \quad g_1 = h(0, y)/F_2(y)|_{y = y_*} \neq 0, \quad g_2 = dF_4/dy|_{y = y_*} \neq 0$$

is obtained in the variables $\eta = y - y_*$.

It remains to show that system (5.5) can be stabilized to the state (5.8). In order to do this, we use the Lyapunov function

$$E_2(y, \dot{y}) = \frac{1}{2}h(0, y)\dot{y}^2 + B_2(y) - B_2(y_*), \quad B_2(y) = \int_0^y F_2(\theta)[F_4(\theta) + \chi]d\theta$$

Just as the relations

$$dB_2/dy = F_2(\theta)[F_4(\theta) + \chi] = 0, \quad d^2B_2/dy^2 = F_2(y)dF_4/dy > 0$$

are satisfied when $y = y_*$ by virtue of the equalities (5.9) and (5.7), so the inequality $B_2(y) \geq B_2(y_*)$ holds and this means that the Lyapunov function has the form

$$B_3(\eta) = B_2(\eta + y_*) - B_2(y_*)$$

By virtue of Eq. (5.5), the time derivative of the function $E_2(y, \dot{y})$

$$dE_2/dt = F_2(y)\dot{y}[u_5 - \dot{y}^2 F_3(y)/2]$$

and, when account is taken of the first relation of (5.7), it will satisfy the inequality $dE_2/dt \leq 0$ if we take, for example,

$$u_5 = -2\pi^{-1} a_5 \operatorname{arctg}[F_2(y)\dot{y}] \tag{5.10}$$

Here, $F_2(y) \neq 0$, and the particular solution $\dot{y} \equiv 0$ accompanying the free motion ($u_5 \equiv 0$) is only possible for $y = y_*$ by virtue of the uniqueness of the equilibrium state. Finally, if we take the number χ is advance (and, also, the value of y_* calculated by virtue of equality (5.9)) to be sufficiently small, then the compactness of the sets $H_c(B_3(\eta))$ will be guaranteed in view of the homeomorphism of these sets with their quadratic approximations

$$B_3(\eta) \approx k\eta^2/2, \quad k = F_2(y)dF_4/dy|_{y = y_*}$$

By virtue of the Barbashin - Krasovskii theorem, the above properties ensure the asymptotic stability of solution (5.8) with control (5.10), which it was required to prove.

Remark 6. Suppose system (1.1), (1.2), (1.4) with two degrees of freedom is stabilizable and, in the equilibrium state, the reaction of unilateral constraint (4.3) is equal to zero. If the condition $r > 0$ is satisfied for the equations in the form of (4.2) and the functions defined in the form (5.6) satisfy condition (5.7), then the system is globally controllable.

Actually, the system then has a globally accessible curve (5.3), (5.8) which, according to the corollary from Assertion 3, means global controllability.

Example 6. Two point masses with masses m_1 and m_2 (small balls within a smooth tube) move in a gravitational field in the vertical plane xOy along an infinite curve which has the form of a catenary $y = l\,\mathrm{ch}(x/l)$.

A control force u, which is collinear with the velocity, is applied to the point m_1, and $|u \leq a$, where a is a specified number. The curved coordinates q_1 (of the first point) and q_2 (of the second point) are measured from the lower equilibrium position. In view of the constraint $q_1 \geq q_2$, collisions between the balls are possible which we assume to be absolutely elastic. In the dimensionless variables and dimensionless time

$$q_i' = \frac{q_i}{l}(i = 1, 2), \quad \mu = \frac{m_2}{m_1}, \quad u' = \frac{u}{m_1 g}, \quad a' = \frac{a}{m_1 g}, \quad t' = t\sqrt{\frac{g}{l}}$$

we obtain (omitting the prime) the reduced Lagrange function

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \dot{q}_1^2/2 + \mu\dot{q}_2^2/2 - B(\mathbf{q}), \quad \mathbf{q} = (q_1, q_2)^T$$

The potential energy

$$B(q_1, q_2) = \sqrt{q_1^2 + 1} - 1 + \mu(\sqrt{q_2^2 + 1} - 1) \tag{5.11}$$

is a connected Lyapunov function in $P = \{\mathbf{q} \in \mathbf{R}^2 : q_1 - q_2 \geq 0\}$ since the conditions of Assertion 1 are satisfied: here, $Q_B(\partial P)\backslash\mathbf{0} = \emptyset$ and the set $\psi_v(\varepsilon)$ is the interval $[-c_0, c_0]$, $c_0 = \sqrt{2\varepsilon/[(1+\mu)^2 - 2\varepsilon]}$. In the equations of motion

$$\begin{aligned} &\ddot{q}_1 + q_1/\sqrt{q_1^2 + 1} = u \\ &\ddot{q}_2 + q_2/\sqrt{q_2^2 + 1} = 0, \quad q_1 - q_2 \geq 0 \end{aligned} \tag{5.12}$$

interference of the variable is only achieved by means of impacts. The sets $H_c(B(\mathbf{q}))$ are compact and, in the case of the free motion ($u \equiv 0$), the system does not allow of the particular solution $\dot{q}_1 \equiv 0$ (apart from the equilibrium state $\mathbf{q} \equiv \mathbf{0}$), since there are no complete trajectories $q_2(t)$ in the domain $q_2 \leq 0$, apart from $q_2 \equiv 0$. System (5.12) is therefore stabilizable with respect to an input u (by virtue of Assertion 2).

In the ε-neighbourhood of the point $(\mathbf{0}, \mathbf{0})$, we change to the variables[5]

$$s = q_1 - q_2, \quad y = (1+\mu)q_2 + s$$

where the momentum $\dot{y} = \partial T/\partial \dot{q}_2$ is calculated after substituting $\dot{q}_1 = \dot{s} + \dot{q}_2$ into the expression for T. The kinetic energy acquires a canonical structure with the coefficients $\lambda = \mu/(1+\mu)$.

The unsmooth substitution[5]

$$s = |x|, \quad \dot{s} = \dot{x}\operatorname{sign}x$$

and the choice of the control in the form of (5.1) leads to Eqs. (5.2) and (5.5) where $r = 1$, $\rho = 1$ and $\partial h/\partial|x| \equiv 0$, and the potential energy $B(|x|, y)$ is obtained by substituting the expressions

$$q_1 = (y + \mu|x|)/(1+\mu), \quad q_2 = (y - |x|)/(1+\mu)$$

into equality (5.11). As a result, the functions

$$F_2(y) \equiv 1, \quad F_3(y) \equiv 0, \quad F_4(y) = y/\sqrt{y^2 + (1+\mu)^2}$$

calculated according to equalities (5.6), satisfy conditions (5.7).

By virtue of Remark 6, the system consisting of two balls in a tube which has been considered is globally controllable, that is, after a finite time it can be transferred, in the space $(\mathbf{q}, \dot{\mathbf{q}})$ (when the condition $q_1 - q_2 \geq 0$ is satisfied), from any initial state into any required state by an admissible control $|u| \leq a$, where a is a specified number.

Finally, we note that question of stability and control in mechanical systems with unilateral constraints have attracted many investigators in recent years.[13,14]

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References

1. Lee EB, Markus L. *Foundations of Optimal Control Theory*. New York: Wiley; 1967.
2. Kayumov OR. On the global controllability of certain Lagrangian systems. *Izv Akad Nauk SSSR MTT* 1986;**6**:16–23.
3. Boretskii IF, Kayumov OR. Globally controllable systems of solid rigid. *Prikl Mat Mekh* 1998;**62**(3):405–12.
4. Kayumov OR. Globally controllable systems of rigid bodies with several stable states of rest. *Prikl Mat Mekh* 2002;**66**(5):775–81.
5. Zhuravlev VF. Mechanics of systems with unilateral constraints. *Uspekhi Mekhaniki* 1989;**12**(2):37–69.
6. Appell P. *Traite de mecanique rationelle T. II*. Paris: Gauthier-Villars; 1953.
7. Ivanov AP. The problem of constrained impact. *Prikl Mat Mekh* 1997;**61**(3):355–68.
8. Kozlov VV. A constructive method for establishing of a theory of systems with non- retaining constraints. *Prikl Mat Mekh* 1988;**52**(6):883–94.
9. Kayumov OR. Asymptotic stability in the large in systems with a cylindrical phase space. *Izv Vuzov Matematika* 1987;**10**:61–3.
10. Shilov GYe. *Mathematical Analysis. Functions of Several Real Variables*. Moscow: Nauka; 1972.
11. Filippov AF. *Differential Equations with Continuous Right-hand Side*. Dordrecht: Kluwer; 1988.
12. Ivanov AP, Markeyev AP. The dynamics of systems with unilateral constraints. *Prikl Mat Mekh* 1984;**48**(4):632–6.
13. Brogliato B, Niculescu S, Orhant P. On the control of finite-dimensional mechanical systems with unilateral constraints. *IEEE Trans Automat Contr* 1997;**42**(2):200–15.
14. Brogliato B. Some perspectives on the analysis and control of complemenltary systems. *IEEE Trans Automat Contr* 2003;**48**(6):918–35.

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